

MAXIMUM LIKELIHOOD CRITERIA FOR BINARY ASYMMETRIC CHANNELS

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ABSTRACT. This work concerns with the n -fold binary asymmetric channels (BAC^n). An equivalence relation between two channels can be characterized by both having the same decision criterion when maximum likelihood is considered. We introduce here a function \mathcal{S} (the BAC-function) such that the parameters (p, q) of the binary channel which determine equivalent channels belong to certain region delimited by its level curves. Explicit equations determining these regions are given and the number of different BAC^n classes is determined. A discussion on the size of these regions is also presented.

1. INTRODUCTION

The Binary Asymmetric Channel $BAC(p, q)$ is the discrete memoryless channel with binary alphabet and transition probability given by $\Pr(1|0) = p, \Pr(0|0) = 1 - p, \Pr(0|1) = q$ and $\Pr(1|1) = 1 - q$, where $\Pr(x|y)$ denote the probability of receiving x if y was sent. Without loss of generality we will assume $p \leq q$ and $p + q < 1$ (see discussion in Section 4).

There are two extreme cases of binary asymmetric channels: the binary symmetric channels $BSC(p) := BAC(p, p)$ with $0 < p < \frac{1}{2}$ and the Z-channels $Z(q) := BAC(0, q)$ with $0 < q < 1$. We may say that all binary symmetric channels are equivalent from the point of view of encoding and decoding, and the same can be said regarding the Z-channels. To be more precise, let W_1 and W_2 be two binary symmetric channels or two Z-channels and let $Pr_W^e(C)$ denotes the error probability of the code $C \subseteq \mathbb{F}_2^n$ according to the channel W . On the encoding part, given two n -block codes $C_1, C_2 \subseteq \mathbb{F}_2^n$ (for some $n \geq 1$) we have that $Pr_{W_1}^e(C_1) \leq Pr_{W_1}^e(C_2)$ if and only if $Pr_{W_2}^e(C_1) \leq Pr_{W_2}^e(C_2)$. This means that if C_1 has smaller error probability than C_2 with respect to a specific BSC (Z-channel) the same happens for all BSCs (Z-channels). On the decoding part, for every n -block code $C \subseteq \mathbb{F}_2^n$ we have $\arg \max_{c \in C} \Pr_{W_1}(x|c) = \arg \max_{c \in C} \Pr_{W_2}(x|c)$ for all $x \in \mathbb{F}_2^n$. This means that the maximum likelihood decoding does not depend on the BSC (or Z-channel) considered. These characteristics of the extreme channels does not hold for general binary asymmetric channels. That is, on the encoding part, there are channels W_1 and W_2 neither of them binary symmetric channel nor Z-channel and codes $C_1, C_2 \in \mathbb{F}_2^n$ verifying $Pr_{W_1}(C_1) \leq Pr_{W_1}(C_2)$ and $Pr_{W_2}(C_1) > Pr_{W_2}(C_2)$. On the decoding part, for every fixed $n \geq 1$, there exist channels W_1 and W_2 neither of them binary symmetric channel nor Z-channel and a code $C \subset \mathbb{F}_2^n$ such that $\arg \max_{c \in C} \Pr_{W_1}(x|c) \neq \arg \max_{c \in C} \Pr_{W_2}(x|c)$ for some $x \in \mathbb{F}_2^n$, consequently these channels determine different MLD criteria. In this

paper we focus on the problem of determining when two binary asymmetric channels determine the same MLD criteria, when restricted to n -block codes. A simple necessary and sufficient condition for two channels $BAC(p, q)$ and $BAC(p', q')$ to determine the same decoding criterion is given. This condition (which depends on n) induces an equivalence relation on the parameter space and we obtain an explicit formula for the number of non-equivalent channels.

We remark that the interest in binary asymmetric channels has increased due to applications in flash memories [5, 6, 10, 18] and as models in other areas, such as neuroscience [3]. Most of the work developed on binary asymmetric channels has focused on coding properties [4, 14] and on the design of codes with given properties, either for general asymmetric channels [2, 4, 9, 11, 17, 19] or for the Z -channel [12, 13, 15]. The substantial efforts devoted to the encoding perspective contrasts with the situation concerning the decoding aspect, the focus of this work.

This paper is organized as follows: In Section 2 we introduce an equivalence relation between channels in such a way that equivalent channels determine equal decoding criteria when MLD is considered and discuss some properties of this relation. We consider for each (fixed) $n \geq 2$ the above equivalence relation restricted to the parametric family of channels $BAC^n(p, q)$. A description of the regions in the parameter space determined by this equivalence relation and the number of such regions is provided, for each $n \geq 2$, in Section 3. In Section 4 we discuss some properties of the BAC-function and the areas of the regions determined by its level curves, which are related to the probability of a given channel to have some prescribed admissible criterion.

2. MLD-EQUIVALENT CHANNELS

We start by introducing an equivalence relation (depending on $n \in \mathbb{N}$) that characterizes when two channel determine the same decoding criteria if MLD is considered over all n -block codes.

Let W be a discrete memoryless channel with input and output alphabet \mathcal{X} and $n \in \mathbb{N}$. For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathcal{X}^n$ we denote by $\Pr_W(x|y) := \prod_{i=1}^n \Pr_W(x_i|y_i)$ the probability of receiving x if y was sent through the channel W . When $W = BAC(p, q)$ we denote this probability by $\Pr_{(p,q)}(x|y)$ and if the channel is understood we denote it simply $\Pr(x|y)$.

Definition 2.1. Let $W_i : \mathcal{X} \rightarrow \mathcal{X}$, for $i = 1, 2$, be two discrete memoryless channels with input and output alphabet \mathcal{X} and $n \in \mathbb{N}$. We say that these channels are n -equivalent (denoted by $W_1 \sim_n W_2$) if for every n -block code $C \subseteq \mathbb{F}_2^n$ and every word $x \in \mathcal{X}^n$, we have

$$\arg \max_{y \in C} \Pr_{W_1}(x|y) = \arg \max_{y \in C} \Pr_{W_2}(x|y)$$

where we interpret $\arg \max$ as returning lists of size at least 1.

We remark that two channels W_1 and W_2 are n -equivalent if and only if the maximum likelihood decoding on W_1 coincides with the maximum likelihood decoding on W_2 when restricted to n -block codes.

Let $W : \mathcal{X} \rightarrow \mathcal{X}$ be a discrete memoryless channel. The n -fold associated channel is a channel $W^n : \mathcal{X}^n \rightarrow \mathcal{X}^n$ whose transition probability is given by

$$\Pr_{W^n}(x|y) = \prod_{i=1}^n \Pr_W(x_i|y_i),$$

for $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathcal{X}^n$. In particular the n -fold channel associated with $W = \text{BAC}(p, q)$ is denoted by $\text{BAC}^n(p, q)$. We identify the set $\{0, 1, \dots, 2^n - 1\}$ with the elements of \mathbb{F}_2^n via its binary expansion. Under this identification, we associate with the n -fold binary asymmetric channel $\text{BAC}^n(p, q)$ a matrix $M_n(p, q) = (m_{xy})_{0 \leq x, y < 2^n}$ whose xy -entry is equal to

$$m_{xy} = \Pr(x|y) = \prod_{i=1}^n \Pr_{(p,q)}(x_i|y_i),$$

where x_i and y_i is the i -th bit of x and y respectively. For example, the matrix $M_5(p, q)$ is a 32×32 matrix whose entry in the row $9 = 01001_2$ and column $29 = 11101_2$ is $m_{9,29} = \Pr(01001|11101) = \Pr(0|0)\Pr(0|1)^2\Pr(1|1)^2 = (1-p)q^2(1-q)^2$.

With this terminology, we remark that two channels W_1 and W_2 are n -equivalent if and only if their associated n -fold channels W_1^n and W_2^n are 1-equivalent. When two channels are 1-equivalent we say that they are MLD-equivalent.

When considering MLD-equivalence, we need to compare the entries in each line of the channel matrix and it is useful to replace it by a simpler matrix. In [7], the authors substituted a matrix M by a matrix \tilde{M} with entries $\tilde{M}_{ij} = k$ if M_{ij} is the k -th largest element (allowing ties) of the j -th column. In order to determine whether two channels are equivalent we adopt a different, but similar substitution.

Definition 2.2. Let M be an $N \times N$ real matrix. The *ordered form* of M is the integer matrix M^* defined by $M_{ij}^* = \#\{k : 1 \leq k \leq N, M_{ik} < M_{ij}\}$.

The ordered form $[W]^*$ of the channel matrix $[W]$ will be used to characterize the equivalent n -fold BACs. We observe that $M^{**} = M^*$ for all $M \in \mathcal{M}_N(\mathbb{R})$. The following equivalences are immediate consequence of the definition of $[W]^*$.

Proposition 2.3. Let $W_1, W_2 : \mathcal{X} \rightarrow \mathcal{X}$ be two discrete channels with input and output alphabet $\mathcal{X} = \{x_1, \dots, x_N\}$. Let M_1 and M_2 denote the matrices of the n -fold channels W_1^n and W_2^n respectively. The following assertions are equivalent.

- i) The channels W_1 and W_2 are n -equivalent.
- ii) $\Pr_{W_1^n}(x|y) \leq \Pr_{W_1^n}(x|z) \Leftrightarrow \Pr_{W_2^n}(x|y) \leq \Pr_{W_2^n}(x|z)$ for all $x, y, z \in \mathcal{X}^n$.
- iii) $M_1^* = M_2^*$.

Let $W_1, W_2 : \mathcal{X} \rightarrow \mathcal{X}$ be two discrete channels and $\xi \in \mathcal{X}$. Considering the map $\mathcal{X}^n \rightarrow \mathcal{X}^{n+1}$ given by $x \mapsto (x, \xi)$, it is easy to check that $W_1 \underset{n+1}{\sim} W_2$ implies $W_1 \underset{n}{\sim} W_2$. When $W_1 \underset{n}{\sim} W_2$ for all $n \geq 1$ we say that these channels are ∞ -equivalent (in this case we denote $W_1 \underset{\infty}{\sim} W_2$).

We consider the parameter space $\mathcal{T} = \{(p, q) \in \mathbb{R}^2 : 0 \leq p \leq q, p+q < 1\} \setminus \{(0, 0)\}$ for the binary asymmetric channel and let $n \geq 1$ be a fixed integer or $n = \infty$. The n -equivalence for BACs induces an equivalence relation in \mathcal{T} : $(p, q) \underset{n}{\sim} (p', q')$ if and

only if $BAC(p, q) \sim_n BAC(p', q')$. We denote by $\Delta_n = \mathcal{T} / \sim_n$, the set of equivalence classes and by $\pi_n : \mathcal{T} \rightarrow \Delta_n$ the projection that associates (p, q) to its equivalence class (i.e. $\pi_n(p, q) = \{(p', q') \in \mathcal{T} : (p', q') \sim_n (p, q)\}$). The main result of this paper is a complete description of the quotient set Δ_n .

Definition 2.4. A decision criterion of order n for the BAC is an equivalence class $\mathcal{A} \in \Delta_n$ (in particular \mathcal{A} is a subset of \mathcal{T}). An n -stable decision criterion \mathcal{A} is a decision criterion of order n for the BAC which is an open set of \mathcal{T} and an n -unstable decision criterion is a decision criterion of order n for BAC with no interior points.

Remark 2.5. If \mathcal{A} is a stable decision criterion for the n -fold BAC and $(p, q) \in \mathcal{A}$ then maximum likelihood decoding on $BAC^n(p, q)$ remains the same under small perturbation of the parameter (p, q) .

Definition 2.6. A point $(p, q) \in \mathcal{T}$ is n -stable if (p, q) is an interior point of $\pi_n((p, q))$ and n -unstable otherwise. The n -stable region (denoted by \mathcal{R}_n^{st}) is the set of all n -stable points and the n -unstable region (denoted by \mathcal{R}_n^{un}) is the set of all n -unstable points.

We will prove later that every decision criterion is either stable or unstable, that is, if a criterion contain an n -stable point then all its points are n -stable. We conclude this section discussing how the parameter space \mathcal{T} decomposes into different n -equivalence classes for $n \leq 5$.

For $n = 1$ we have $\mathcal{X} = \{0, 1\}$ and the matrix of $BAC^1(p, q)$ is given by $M_1(p, q) = \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix}$, thus its ordered form $M_1(p, q)^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does not depend on (p, q) and we have only one criterion, which is stable.

For $n = 2$ we have $\mathcal{X} = \{00, 01, 10, 11\}$ and the matrix $M_2(p, q)$ of $BAC^2(p, q)$ (with $(p, q) \in \mathcal{T}$) is given by

$$\begin{pmatrix} (1-p)^2 & (1-p)q & (1-p)q & q^2 \\ (1-p)p & (1-p)(1-q) & pq & q(1-q) \\ (1-p)p & pq & (1-p)(1-q) & q(1-q) \\ p^2 & p(1-q) & p(1-q) & (1-q)^2 \end{pmatrix}.$$

In this case the ordered form depends on (p, q) :

- If (p, q) is an interior point of \mathcal{T} the ordered form is given by $M_2(p, q)^* =$

$$\begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 3 \end{pmatrix};$$

- If $p = 0$ (and hence $q > 0$, since $(0, 0) \notin \mathcal{T}$) we obtain the ordered form

$$M_2(p, q)^* = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix};$$

- If $p = q$ we obtain $M_2(p, q)^* = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix}.$

In this case we have three different decision criteria, one stable and two unstable.

For $n = 3, 4$ and 5 we started with some simulations using the software SAGE [16]. We considered a set A of $28900 = 170^2$ points uniformly distributed on \mathcal{T} and calculated the ordered form $M_n(p, q)^*$ of each $(p, q) \in A$.

For $n = 3$ we observe two criteria, \mathcal{B} and \mathcal{R} , and by coloring the points in these regions by blue and red respectively we obtain the picture showing in Figure 1. As

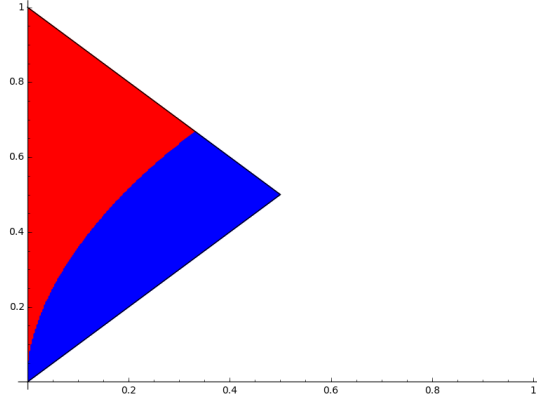


FIGURE 1. Each color corresponds to a stable decision criterion for the 3-fold BAC.

we will prove later (Theorem 3.10), there are two stable criteria \mathcal{B} and \mathcal{R} (the connected components of the stable region \mathcal{R}_3^{st}) and there are three unstable decision criteria corresponding to the curves $p = 0$ (the Z -channel), $p = q$ (the BSC) and the curve that separates the two connected components of the stable region.

We proceed similarly with the cases $n = 4$ and $n = 5$, observing three decision criteria for $n = 4$ (Figure 2) and five decision criteria for $n = 5$ (Figure 3). They correspond to the stable decision criteria and the curves separating these regions correspond to the four and six different unstable decision criteria for $n = 4$ and $n = 5$ respectively (see Theorem 3.10 in the next section).

3. DETERMINING THE DECISION CRITERIA FOR THE n -FOLD BAC

We start introducing a function which plays a fundamental role in describing the regions which determine the decision criteria for the BAC. This function also induces a natural distance between binary asymmetric channels (see Section 4 for more details).

Definition 3.1. Let $S : \mathcal{T} \rightarrow [0, 1]$ be the function given by

$$S(p, q) = \frac{\ln(1-p) - \ln(q)}{\ln(1-q) - \ln(p)},$$

if $p \neq 0$, and $S(p, q) = 0$ if $p = 0$. We refer to this function as the BAC-function.

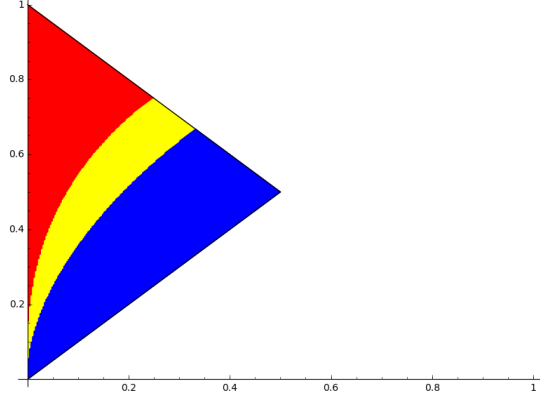


FIGURE 2. Each color corresponds to a stable decision criterion for the 4-fold BAC.

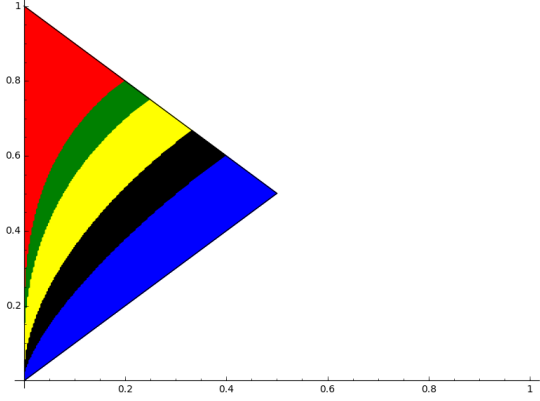


FIGURE 3. Each color corresponds to a stable decision criterion for the 5-fold BAC.

It is easy to check that in fact the image of S is contained in the interval $[0, 1]$ where the values 0 and 1 are attained by the extremes cases $p = 0$ and $p = q$, respectively. Besides, this function is continuous in the connected set \mathcal{T} and therefore we have $S(\mathcal{T}) = [0, 1]$.

Let a and b be integers with $0 \leq a \leq b$. For $(p, q) \in \mathcal{T}$ we have

$$(1) \quad p^a(1-p)^b \geq q^b(1-q)^a \Leftrightarrow S(p, q) \geq \frac{a}{b},$$

where equality corresponds to equality. The next lemma is a direct consequence of the above relation.

Lemma 3.2. *Let a and b be natural numbers with $a \leq b$, $a + b \leq n$ and $\eta = n - (a + b) \geq 0$. Consider the words $x = 1^{a+\eta}0^b, y = 0^n, z = 0^\eta 1^{a+b} \in \mathbb{F}_2^n$. Then $S(p, q) \leq a/b$ if and only if $\Pr(x|y) \leq \Pr(x|z)$, where equality corresponds*

to equality. In particular, if $S(p_0, q_0) \leq a/b$ and $S(p_1, q_1) > a/b$ the channels $BAC(p_0, q_0)$ and $BAC(p_1, q_1)$ are not n -equivalent.

The above lemma establishes a condition for the n -fold channels $BAC^n(p_0, q_0)$ and $BAC^n(p_1, q_1)$ not being MLD-equivalent. When we consider p and q as variables, the entries of the transition matrix $M_n(p, q)$ of the channel $W = BAC^n(p, q)$ are polynomials in the variables p and q . In fact, the entry of $M_n(p, q)$ corresponding to the conditional probability $\Pr(x|y)$ ($x, y \in \mathbb{F}_2^n$) is equal to the polynomial $f(p, q) = p^a(1-p)^b q^c(1-q)^d$ where a, b, c and d correspond to the number of indices i for which (x_i, y_i) is equal to $(1, 0), (0, 0), (0, 1)$ and $(1, 1)$, respectively. In particular we have $a + b + c + d = n$ and the Hamming weight of x is equal to $a + d$, which we will refer also as the weight of f and denote by $\omega(f)$. We remark that two polynomials in the same row of $M_n(p, q)$ have necessarily the same weight. By the previous consideration we define the following sets of bivariate polynomials:

- $\mathcal{P}^n = \{f(p, q) = p^a(1-p)^b q^c(1-q)^d : a, b, c, d \geq 0, a + b + c + d = n\}$.
- $\mathcal{P}_k^n = \{f \in \mathcal{P}^n : \omega(f) = k\}$ for $0 \leq k \leq n$.

To determine the decision criterion of an n -fold BAC we need to do comparisons only between values in the same row of its transition matrix what means comparisons of values of polynomials belonging to \mathcal{P}_k^n for some k , $0 \leq k \leq n$. The next lemma describes the stable region in terms of these sets.

Lemma 3.3. *The n -stable region \mathcal{R}_n^{st} is given by*

$$\mathcal{R}_n^{st} = \bigcap_{k=0}^n \bigcap_{\substack{f, g \in \mathcal{P}_k^n \\ f \neq g}} \{x \in \mathcal{T} : f(x) \neq g(x)\}.$$

Proof. We denote by $\widehat{\mathcal{R}}_n^{st}$ the set on the right side of the above equality. This set is open in \mathcal{T} (since it is a finite intersection of open sets in \mathcal{T}), therefore if $(p_0, q_0) \in \widehat{\mathcal{R}}_n^{st}$ there is a ball B_0 with center at this point such that $B_0 \cap \mathcal{T} \subseteq \widehat{\mathcal{R}}_n^{st}$. Since $B_0 \cap \mathcal{T}$ is connected, the signal of $f(p, q) - g(p, q)$ does not depends on $(p, q) \in B_0 \cap \mathcal{T}$ (whenever $f, g \in \mathcal{P}_k^n$ for some k) and the same occurs with their associated decision criteria, so $\widehat{\mathcal{R}}_n^{st} \subseteq \mathcal{R}_n^{st}$. To prove the other inclusion we suppose by contradiction that there exists $(p, q) \notin \widehat{\mathcal{R}}_n^{st}$ verifying $(p, q) \in \mathcal{R}_n^{st}$. Then, there exists two distinct polynomials $f, g \in \mathcal{P}_k^n$ for some $k : 0 \leq k \leq n$ and a ball B_0 centered at (p_0, q_0) such that $f(p_0, q_0) = g(p_0, q_0)$ and every point in $B_0 \cap \mathcal{T}$ determines the same decision criterion, in particular $f(p, q) = g(p, q)$ for all $(p, q) \in B_0 \cap \mathcal{T}$. Since $B_0 \cap \mathcal{T}$ has interior points and two polynomial that coincide in an open set must be equal, we have $f = g$ which is a contradiction. \square

As we will prove next, the main property of the BAC-function from the point of view of this work, is that the curves which separate the regions corresponding to the stable and unstable criteria are level curves of this function associated with rational values. First, we prove some lemmas.

Lemma 3.4. *Let $(p, q) \in \mathcal{T}$ be an n -unstable point for the BAC. Then $S(p, q) = \frac{a}{b}$ where a, b are integers verifying $a \geq 0, b \geq 1, a \leq b, \gcd(a, b) = 1$ and $a + b \leq n$.*

Proof. By Lemma 3.3, if $(p_0, q_0) \in \mathcal{T}$ is an n -unstable point for the BAC then there exists distinct polynomials $f_1, f_2 \in \mathcal{P}_k^n$ for some $k : 0 \leq k \leq n$ such that $f_1(p_0, q_0) = f_2(p_0, q_0)$. We write $f_i(p, q) = p^{a_i}(1-p)^{b_i} q^{c_i}(1-q)^{d_i}$ with $a_i + b_i + c_i + d_i = n$

and $a_i + d_i = k$ for $i = 1, 2$. Without loss of generality we suppose $a_1 \geq a_2$. Let $a := a_1 - a_2 = d_2 - d_1$ and $b := b_1 - b_2 = c_2 - c_1$, then we have

$$\frac{f_1(p, q)}{f_2(p, q)} = \left(\frac{p}{1-q} \right)^a \left(\frac{1-p}{q} \right)^b,$$

or, equivalently,

$$(2) \quad p^a(1-p)^b = \left(\frac{f_1(p, q)}{f_2(p, q)} \right) \cdot q^b(1-q)^a.$$

Evaluating the Equation 2 for $(p, q) = (p_0, q_0)$ and using the relation (1) we have $S(p_0, q_0) = \frac{a}{b}$. By our assumption we have $a \geq 0$, since $S(\mathcal{T}) = [0, 1]$ then $b \geq 1$ and $a + b = a_1 - a_2 + b_1 - b_2 \leq a_1 + b_1 \leq n$; simplifying common factors if necessary we can assume $\gcd(a, b) = 1$. \square

Based on the above result, we introduce the following definition.

Definition 3.5. The weight of a non-negative rational r (denoted by $\omega(r)$), is the sum of its numerator and denominator in the reduced expression of r . An n -critical value for the BAC-function S (where $n \geq 2$) is a rational number $r \in [0, 1]$ with $\omega(r) \leq n$. The set of all n -critical values for S is denoted by \mathcal{D}_n .

Corollary 3.6. If we write the set of the n -critical values for S as $\mathcal{D}_n = \{r_0 = 0 < r_1 < \dots < r_t = 1\}$ and denote by $R_n(r_i) = \{(p, q) \in \mathcal{T} : r_i < S(p, q) < r_{i+1}\}$ then $R_n(r_i) \subseteq \mathcal{R}_n^{st}$ for $0 \leq i < t$.

Lemma 3.7. Let $(p, q) \in \mathcal{T}$ and $M_n(p, q)$ be the transition matrix for the channel $BAC^n(p, q)$ (seeing as an element of \mathbb{R}^{n^2}). The function $\phi : \mathcal{R}_n^{st} \rightarrow \mathbb{R}^{n^2}$ given by $\phi(p, q) = M_n(p, q)^*$ is continuous.

Proof. Let $(p_0, q_0) \in \mathcal{R}_n^{st}$ and $f, g \in \mathcal{P}_k^n$ for some k , $0 \leq k \leq n$ with $f \neq g$. By Lemma 3.3 we have $f(p_0, q_0) \neq g(p_0, q_0)$, then there exists $\epsilon = \epsilon(f, g) > 0$ such that the sign of $f(p, q) - g(p, q)$ does not depend on $(p, q) \in B((p_0, q_0), \epsilon) \cap \mathcal{T}$. If $\epsilon = \min\{\epsilon(f, g) : f, g \in \mathcal{P}_k^n, f \neq g, 0 \leq k \leq n\}$ and denoting by B the ϵ -ball centered at (p_0, q_0) we have that $f(p, q) > g(p, q) \Leftrightarrow f(p_0, q_0) > g(p_0, q_0)$ for all $(p, q) \in B \cap \mathcal{T}$ and for all $f, g \in \mathcal{P}_k^n, f \neq g, 0 \leq k \leq n$. Therefore $M_n(p, q)^* = M_n(p_0, q_0)^*$, so ϕ is locally constant and in particular continuous. \square

Lemma 3.8. Let S be the BAC-function. Then $S^{-1}(I)$ is a connected set for every interval $I \subseteq [0, 1]$.

Proof. Let $\tau \in (0, 1)$ and $g_\tau : [0, \frac{\tau}{2}] \rightarrow [0, 1]$ be the function given by $g_\tau(p) = S(p, \tau - p)$. We affirm that it is increasing (in the variable p). To prove this we consider $s \in [0, 1]$ and the function $f_s(p) = p^s(1-p) - q(1-q)^s$ (where $q = \tau - p$). Since $1 - q > p$ we have $p^{s-1} > (1-q)^{s-1}$ and $(1-q)^s > p^s$, therefore

$$\begin{aligned} f'_s(p) &= s(p^{s-1}(1-p) - q(1-q)^{s-1}) + (1-q)^s - p^s \\ &> s(1-q)^{s-1}(1-p-q) + (1-q)^s - p^s > 0. \end{aligned}$$

Since $f_s(0) = -q(1-q)^s < 0$ and $f_s(\tau/2) = \tau/2(1-\tau/2)((\tau/2)^{s-1} - (1-\tau/2)^{s-1}) > 0$ (because $1-\tau/2 > \tau/2$ and $s-1 < 0$), then there is a unique $p \in (0, \tau/2)$ such that $f_s(p) = 0$, or equivalently, such that $g_\tau(p) = S(p, \tau - p) = s$. Therefore $g_\tau : [0, \frac{\tau}{2}] \rightarrow [0, 1]$ is increasing since it is a continuous bijection with $g_\tau(0) = 0$ and $g_\tau(\tau/2) = 1$. Let $I \subseteq [0, 1]$ be an interval and $(p_0, q_0), (p_1, q_1) \in \mathcal{T}$ be two

points in $S^{-1}(I)$. We denote by $s_i = S(p_i, q_i)$ and $\tau_i = p_i + q_i$ for $i = 0, 1$. Without loss of generality we assume $s_0 \leq s_1$. Since $g_{\tau_0}(p_0) = S(p_0, q_0) = s_0 < s_1$ there exists $t_0 > 0$ such that $g_{\tau_0}(p_0 + t_0) = s_1$ and $g_{\tau_0}(p_0 + t) \in (s_0, s_1)$ for all $t \in (0, t_0)$ (in particular $(p_0 + t, q_0 - t) \in S^{-1}(I)$ for all $t \in [0, t_0]$). Taking the path $\alpha : [0, t_0] \rightarrow \mathcal{T}$ given by $\alpha(t) = (p_0 + t, q_0 - t)$ and β the segment of curve in $S^{-1}(s_1)$ from $(p_0 + t_0, q_0 - t_0)$ to (p_1, q_1) , we have that the concatenation path $\beta * \alpha$ is a path connecting (p_0, q_0) with (p_1, q_1) . Therefore $S^{-1}(I)$ is path-connected and then connected. \square

Lemma 3.9. *Let (p_0, q_0) and (p_1, q_1) be two points in \mathcal{T} . If $S(p_0, q_0) = S(p_1, q_1)$ then $BAC(p_0, q_0) \underset{n}{\sim} BAC(p_1, q_1)$ for all $n \geq 1$.*

Proof. We suppose by contradiction that (p_0, q_0) and (p_1, q_1) are not n -equivalent, so there exists two polynomials $f, g \in \mathcal{P}_k^n$ for some k verifying $f(p_0, q_0) < g(p_0, q_0)$ and $f(p_1, q_1) \geq g(p_1, q_1)$. As in the proof of Lemma 3.4, the polynomials f and g must verify an equation similar to Equation (2):

$$(3) \quad p^a(1-p)^b = \left(\frac{f(p, q)}{g(p, q)} \right) \cdot q^b(1-q)^a$$

for some integers $a \geq 0$ and $b \geq 1$ satisfying $a \leq b$ and $a + b \leq n$. We consider a path α contained in the curve $S^{-1}(r)$ connecting (p_0, q_0) with (p_1, q_1) , since $f/g < 1$ in (p_0, q_0) and $f/g \geq 1$ in (p_1, q_1) , by continuity there exists an intermediate point (p_2, q_2) in α for which $f/g = 1$. Evaluating Equation (3) in $(p, q) = (p_2, q_2)$ and using the relation (1) we have $S(p_2, q_2) = a/b$. Since (p_2, q_2) belongs to α which is contained in the level curve $S^{-1}(r)$ we have $S(p_2, q_2) = r$, then $r = a/b$. Substituting $(p, q) = (p_0, q_0)$ in Equation (3) and using the relation 1 we have $r = S(p_0, q_0) < a/b$ which is a contradiction. Therefore (p_0, q_0) and (p_1, q_1) must be n -equivalent. \square

Now we are ready to state the main result of this paper.

Theorem 3.10. *Let $S : \mathcal{T} \rightarrow [0, 1]$ be the BAC-function*

$$S(p, q) = \frac{\ln(1-p) - \ln(q)}{\ln(1-q) - \ln(p)},$$

for $p \neq 0$, $S(0, q) = 0$ and $\mathcal{D}_n = \{0 = r_0 < r_1 < \dots < r_{t_n} = 1\}$ its set of n -critical values ($n \geq 2$). We consider the level curves $\gamma_i = S^{-1}(r_i)$ for $0 \leq i \leq t_n$ and the regions $R_n(r_i) = \{(p, q) \in \mathcal{T} : r_i < S(p, q) < r_{i+1}\}$ for $0 \leq i < t_n$. Then

$$(4) \quad t_n = 1 + \frac{1}{2} \sum_{k=3}^n \varphi(k)$$

where φ denotes the Euler's totient function and there are exactly t_n stable decision criteria for the n -fold BAC which are given by $\{R_n(r_i) : 0 \leq i < t_n\}$ and exactly $t_n + 1$ unstable decision criteria for the n -fold BAC given by $\{\gamma_i : 0 \leq i \leq t_n\}$.

Proof. Consider \mathcal{T} written as the disjoint union

$$\mathcal{T} = \biguplus_{i=0}^{t-1} R_n(r_i) \uplus \biguplus_{i=0}^t \gamma_i.$$

We have to prove that each $R_n(r_i)$ and each γ_i is a decision criterion (i.e. an equivalent class in Δ_n). If $(p_0, q_0) \in \gamma_i$ for some $i : 0 \leq i \leq t$, by Lemma 3.9

and Lemma 3.2 we have $BAC^n(p_1, q_1) \sim BAC^n(p_0, q_0)$ if and only if $(p_1, q_1) \in \gamma_i$, then each γ_i is a decision criterion. We consider now a point $(p_0, q_0) \in R_n(r_i)$ for some $i : 0 \leq i < t$ and $(p_1, q_1) \in \mathcal{T} \setminus R_n(r_i)$. If $(p_1, q_1) \in \gamma_j$ for some j , since each γ_j is a decision criterion we have $BAC^n(p_1, q_1) \not\sim BAC^n(p_0, q_0)$. Otherwise $(p_1, q_1) \in R_n(r_j)$ for some $j : 0 \leq j < n, j \neq i$ and there exists $r_k \in \mathcal{D}_n$ such that $S(p_0, q_0) < r_k < S(p_1, q_1)$ or $S(p_1, q_1) < r_k < S(p_0, q_0)$. In both cases, by Lemma 3.2 we obtain $BAC^n(p_1, q_1) \not\sim BAC^n(p_0, q_0)$. It only remains to prove that if $(p_1, q_1) \in R_n(r_i)$ the channels $BAC^n(p_1, q_1)$ and $BAC^n(p_0, q_0)$ are MLD-equivalent. We consider the function $\phi : \mathcal{R}_n^{st} \rightarrow \mathbb{R}^{n^2}$ given by $\phi(p, q) = M_n(p, q)^*$ where $M_n(p, q)$ denotes the transition matrix for the channel $BAC^n(p, q)$. By Lemma 3.8 and Lemma 3.4 then $R_n(r_i)$ is a connected set contained in the stable region \mathcal{R}_n^{st} . By Lemma 3.7 the set $\phi(R_n(r_i)) \subseteq \mathcal{M}_n(\mathbb{Z})$ is connected and since $\mathcal{M}_n(\mathbb{Z})$ is discrete, there exists $M \in \mathcal{M}_n(\mathbb{Z})$ such that $\phi(R_n(r_i)) = \{M\}$. Therefore $M_n(p_1, q_1)^* = M_n(p_0, q_0)^* = M$ and $BAC^n(p_1, q_1) \sim BAC^n(p_0, q_0)$.

Since the $R_n(r_i)$ are open sets for $0 \leq i < t_n$ and the sets γ_i have empty interior they correspond to the stable and unstable criteria respectively. To derive the formula for t_n we consider the decomposition into disjoint sets: $\mathcal{D}_n = \biguplus_{k=1}^n \mathcal{D}_n^k$ where $\mathcal{D}_n^k = \{(a, b) \in \mathbb{N}^2 : a \leq b, \gcd(a, b) = 1, a + b = n\}$. We have $\#\mathcal{D}_n^k = 1$ for $k = 1, 2$ and for $k \geq 3$ if $(a, b) \in \mathcal{D}_n^k$ then $a < b$, in this case:

$$\begin{aligned} \#\mathcal{D}_n^k &= \frac{1}{2} \cdot \# \{(a, b) \in \mathbb{N}^2 : \gcd(a, b) = 1, a + b = k\} \\ &= \frac{1}{2} \cdot \# \{(a, b) \in \mathbb{N}^2 : \gcd(a, k) = 1, a + b = k\} \\ &= \frac{1}{2} \cdot \# \{a \in \mathbb{N} : \gcd(a, k) = 1, a \leq k\} = \frac{1}{2} \cdot \varphi(k). \end{aligned}$$

Then, for $n \geq 3$ we have:

$$t_n = \#\mathcal{D}_n - 1 = \sum_{k=1}^n \#\mathcal{D}_n^k - 1 = 2 + \frac{1}{2} \cdot \sum_{k=3}^n \varphi(k) - 1 = 1 + \frac{1}{2} \cdot \sum_{k=3}^n \varphi(k).$$

For $n = 2$ we have $\#\mathcal{D}_2 = \#\{(0, 1), (1, 1)\} = 2$ and the above formula also holds in this case. \square

Corollary 3.11. *Let $(p_0, q_0), (p_1, q_1) \in \mathcal{T}$. The channels $BAC(p_0, q_0)$ and $BAC(p_1, q_1)$ are ∞ -equivalent if and only if $S(p_0, q_0) = S(p_1, q_1)$*

Proof. If $S(p_0, q_0) < S(p_1, q_1)$ there exists a rational number $r \in \mathcal{D}_N$ for some $N \geq 1$ large enough such that $S(p_0, q_0) < r < S(p_1, q_1)$, then the channels $BAC(p_0, q_0)$ and $BAC(p_1, q_1)$ can not be N -equivalent. The converse is consequence of Lemma 3.9. \square

Corollary 3.12. *A point $(p, q) \in \mathcal{T}$ is a stable point for the n -fold channel for all $n \geq 1$ if and only if $S(p, q)$ is irrational.*

Using the formula for the average order for the Euler's totient function φ (see for example Theorem 3.7 of [1]) we obtain the following corollary.

Corollary 3.13. *The number of stable decision criteria for the n -fold BAC grows quadratically with n . More explicitly, it is given by $\frac{3}{\pi^2} \cdot n^2 + O(n \cdot \ln n)$.*

Example 3.14. For $n = 5$ (see Figure 3), we have the set of critical values $\mathcal{D}_5 = \{0, 1/4, 1/3, 1/2, 2/3, 1\}$ which correspond to the level curves of the BAC function describing the unstable sets. Those curves can be seen from left to right according to the order in \mathcal{D}_5 . The five stable regions are the ones bounded by these curves.

Example 3.15. For $n = 9$, we have $t_9 = 29$ decision regions, 15 instable, associated to the critical set $\mathcal{D}_9 = \{0, 1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 1\}$ and 14 stable, situated between the level curves of the BAC function attached to values in \mathcal{D}_9 .

4. FURTHER REMARKS

4.1. The BAC-function and the parameter space for the BACs. To study the different criteria for the n -fold BAC, we choose the parameter space $\mathcal{T} = \{(p, q) \in [0, 1] : p + q < 1, 0 \leq p \leq q\} \setminus \{(0, 0)\}$ and use the BAC-function to describe the regions determined by these criteria. In the first part of this section we discuss what happens when we remove the restriction $p + q < 1$ and $0 \leq p \leq q$ and what is the role of the BAC-function in these cases. We also show how to obtain a natural distance between n -fold BACs in such a way that the BAC-function measures how far a channel is from the binary symmetric channel, in this sense the BAC-function can be interpreted as a measure of the asymmetry of the channel.

We remark first that condition $p + q < 1$ is equivalent to the channel $\text{BAC}^n(p, q)$ being reasonable (in the sense of [8]). More explicitly, for the channel $\text{BAC}^n(p, q)$ we have the following facts (which are consequence of the Equation (2)):

- $p + q < 1 \Leftrightarrow \Pr(x|x) > \Pr(x|y)$ for all $y \neq x$,
- $p + q > 1 \Leftrightarrow \Pr(x|x) < \Pr(x|y)$ for all $y \neq x$,
- $p + q = 1 \Leftrightarrow \Pr(x|x) = \Pr(x|y)$ for all $y \neq x$.

As a consequence, we have that if two channels $\text{BAC}^n(p, q)$ and $\text{BAC}^n(p', q')$ are MLD-equivalent then the sign of $1 - p - q$ and $1 - p' - q'$ is the same. The case $p + q = 1$ corresponds to the completely noisy channel. These channels are all MLD-equivalent among them and are not interesting from the coding/decoding viewpoint. The case $p + q < 1$ can be decomposed into two regions \mathcal{T} and $\mathcal{T}' = \{(p, q) \in [0, 1]^2 : p + q < 1, 0 \leq q \leq p\} \setminus \{(0, 0)\}$ which are symmetric one to the other (via the map $(p, q) \mapsto (q, p)$).

We observe that the BAC-function S can be extended to a function $\hat{S} : \mathcal{T} \cup \mathcal{T}' \rightarrow [0, +\infty]$ defining $\hat{S}(q, p) = 1/S(p, q)$ for $(q, p) \in \mathcal{T}'$ if $pq \neq 0$ and $\hat{S}(p, 0) = +\infty$. This extension is continuous and verifies $\hat{S}(\mathcal{T}') = [1, +\infty]$. By relation (1), $S(p, q) \leq 1$ if and only if $p(1 - p) \leq q(1 - q)$ if and only if $p^{n-1}(1 - p) \leq p^{n-2}q(1 - q)$, therefore for $x = 1^{n-1}0$, $y = 0^n$ and $z = 0^{n-2}1^2$ we have:

- $\Pr(x|y) < \Pr(x|z)$ if $(p, q) \in \mathcal{T}$ with $p \neq q$,
- $\Pr(x|y) > \Pr(x|z)$ if $(p, q) \in \mathcal{T}'$ with $p \neq q$ and
- $\Pr(x|y) > \Pr(x|z)$ if $p = q$.

Thus, we conclude that the triangles \mathcal{T} and \mathcal{T}' have no common criteria decision except for those points corresponding to BSC. Denoting by $M_n(p, q)$ the matrix of $\text{BAC}^n(p, q)$, since the i -th row of $M_n(p, q)$ is just the $(2^n + 1 - i)$ -th row of $M_n(q, p)$ in reverse order, then $\text{BAC}^n(p, q) \sim \text{BAC}^n(p', q')$ if and only if $\text{BAC}^n(q, p) \sim \text{BAC}^n(q', p')$ for all $(p, q), (p', q') \in \mathcal{T}$. By the above consideration we have the following proposition.

Proposition 4.1. *Let $S : \mathcal{T} \cup \mathcal{T}' \rightarrow [0, +\infty]$ be the BAC-function defined as above and $\mathcal{D}_n = \{0 = r_0 < r_1 < \dots < r_{t_n} = 1\}$ its set of n -critical values ($n \geq 2$) where $t_n = 1 + \frac{1}{2} \sum_{k=3}^n \varphi(k)$. There are exactly $2t_n$ stable decision criteria for the $\text{BAC}^n(p, q)$ with $(p, q) \in \mathcal{T} \cup \mathcal{T}'$ and exactly $2t_n - 1$ unstable decision criteria. The stable criteria are given by $R_n(r_i) = \{(p, q) \in \mathcal{T} : r_i < S(p, q) < r_{i+1}\}$ and $R_n(r_i^{-1}) = \{(p, q) \in \mathcal{T} : r_{i+1}^{-1} < S(p, q) < r_i^{-1}\}$ for $0 \leq i < t_n$. The unstable criteria are given by the level curves $S^{-1}(r_i)$ and $S^{-1}(r_i^{-1})$ for $0 \leq i \leq t_n$.*

The function S is constant when restricted to a criterion $\mathcal{A} \subseteq \mathcal{T} \cup \mathcal{T}'$, thus it defines a injective function $S : \Delta_n \rightarrow [0, +\infty]$ such that $S(\mathcal{A}) := S(p, q)$ for any $(p, q) \in \mathcal{A}$. If Δ_n^* denotes the set of all criteria for the n -fold BAC except for those corresponding to the Z -channels (i.e. when $pq = 0$), then the function $d : \Delta_n^* \times \Delta_n^* \rightarrow [0, +\infty)$ given by $d(\mathcal{A}, \mathcal{B}) = |\ln S(\mathcal{A}) - \ln S(\mathcal{B})|$ defines a metric in the criteria space for the n -fold BAC channel. In particular if \mathcal{B} denotes the criterion corresponding to the BSC then $d(\mathcal{A}, \mathcal{B}) = |\ln S(\mathcal{A})|$ can be interpreted as a measure of how asymmetric a channel is.

Since the ordered form of the matrix associated with the completely noisy channels ($p + q = 1$) is the null matrix (because in this case $\Pr(x|y) = p^{w_H(x)}(1 - p)^{n - w_H(x)}$ depends only on the Hamming weight of x and not on y) and this is the only situation for which it happens, then the points in the line $p + q = 1$ correspond to a single criterion when considering n -fold BAC's with $(p, q) \in [0, 1]^2$. We also saw that points with $p + q < 1$ and $p + q > 1$ can not be equivalent (since the first corresponds to reasonable channels and the last not). Let $T^- = \mathcal{T} \cup \mathcal{T}' = \{(p, q) \in [0, 1]^2 : p + q < 1\} \setminus \{(0, 0)\}$ and $T^+ = \{(p, q) \in [0, 1]^2 : p + q > 1\} \setminus \{(1, 1)\}$. The involution $\phi(p, q) = (1 - q, 1 - p)$ maps the triangle T^- into T^+ and the curves $\gamma_{a/b} : p^a(1 - p)^b = q^b(1 - q)^a$ into itself. Moreover, the i -th row of $M_n(p, q)$ and $M_n(1 - q, 1 - p)$ are the same but in reverse order. Therefore for $(p, q), (p', q') \in T^+$, the channels $\text{BAC}^n(p, q)$ and $\text{BAC}^n(p', q')$ are MLD-equivalent if and only if $\text{BAC}^n(1 - q, 1 - p)$ and $\text{BAC}^n(1 - q', 1 - p')$ are MLD-equivalent. We conclude that if we consider n -fold BAC's with parameters $(p, q) \in [0, 1]^2$, the stable criteria are the regions bounded by the edges of the square $[0, 1]^2$ and the curves $\gamma_{a/b}$ with a/b positive rational number with $a + b \leq n$ ($a, b \in \mathbb{Z}$). We also remark that the level curves $\gamma_{a/b}$ contain the line $x + y = 1$ but if we divide the equation that defines it by $x + y - 1$ we obtain irreducible curves $\hat{\gamma}_{a/b}$ (see Figure 4).

4.2. The most probable n -fold BACs are those next to the BSC. By the previous discussion, without loss of generality we can restrict our parameter space to $\mathcal{T} = \{(p, q) \in [0, 1]^2 : p + q < 1, p \leq q\} \setminus \{(0, 0)\}$. Let $\mathcal{A} \subseteq \mathcal{T}$ be a criterion for the n -fold BAC. If we choose a point $(p, q) \in \mathcal{T}$ uniformly at random and consider the channel $W = \text{BAC}^n(p, q)$, the probability $\Pr(W \sim \mathcal{A}) = 4 \cdot a(\mathcal{A})$ where $a(\mathcal{A})$ is the area of the region corresponding to the criterion \mathcal{A} . In this sense, the area of a given criterion for the n -BAC is a measure of how probably this criterion is to be chosen (assuming uniform distribution). By Theorem 3.10, $\mathcal{A} = \{(p, q) \in \mathcal{T} : r_0 < S(p, q) < r_1\} = A(r_1) - A(r_0)$ where r_1 and r_0 are consecutive rational numbers in \mathcal{D}_n and $A(r)$ is the area of the region $R_r = \{(p, q) \in \mathcal{T} : 0 < S(p, q) < r\}$. This

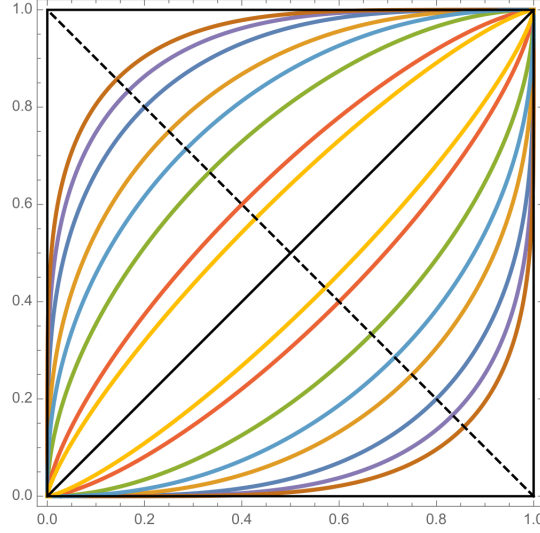


FIGURE 4. Decision regions for the 7-fold BAC with (p, q) in the unit square: the line $p + q = 1$ (dotted) and the curves $\hat{\gamma}_q$ and $\hat{\gamma}_{q^{-1}}$ for $q = 1/6$ (brown), $1/5$ (violet), $1/4$ (blue), $1/3$ (orange), $2/5$ (sky blue), $1/2$ (green), $2/3$ (red), $3/4$ (yellow) and 1 (black). Curves corresponding to reciprocal values have the same color.

area is equal to

$$A(r) = \iint_{R_r} 1 \, dp \, dq.$$

Let $r = a/b$ where a and b are coprime positive integers with $a < b$. Applying the change of variable formula for double integrals with $p = \frac{u-1}{uv-1}$, $q = \frac{v-1}{uv-1}$ and after some calculations we obtain:

$$(5) \quad A(r) = \int_0^1 \frac{b(x^a - 1)^2 x^{b-1}}{2(x^{a+b} - 1)^2} dx.$$

Since $x = 1$ is a zero of order 2 of the numerator, the integral is a proper integral. In some cases, a primitive for the integrand can be calculated explicitly, for example when $r = 1/2$ and $r = 1/3$ obtaining $A(1/2) = \frac{1}{3} - \frac{\sqrt{3}\pi}{27}$ and $A(1/3) = \frac{3}{8} - \frac{3\pi}{32}$. In the other cases we have used the software Wolfram Mathematica [20] to calculate the integral (5) numerically after some reductions.

In particular, the stable criterion for the n -fold BAC nearest the n -fold BSC (denoted by $\mathcal{A}_{Q_S}^n$) is given by the criterion corresponding to the channels $\text{BAC}^n(p, q)$ for $r_n := \frac{2n-3-(-1)^n}{2n+1-(-1)^n} < S(p, q) < 1$ (which we refer as n -fold quasi-symmetric channels). Clearly $a(\mathcal{A}_{Q_S}^n) \rightarrow 0$ when $n \rightarrow \infty$ (since $r_n \rightarrow 1$). In the next table we show the percentages (rounded to the nearest integer) represented by the different stable criteria for the n -fold BAC, $3 \leq n \leq 7$.

| | 0 | $\frac{1}{6}$ | $\frac{1}{5}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{2}{5}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{3}{4}$ | 1 |
|---------|----|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---|
| $n = 3$ | 53 | | | | | | 47 | | | |
| $n = 4$ | 32 | | | | 21 | | 47 | | | |
| $n = 5$ | 22 | | | 11 | 21 | | 18 | 29 | | |
| $n = 6$ | 16 | | 6 | 11 | 21 | | 18 | 29 | | |
| $n = 7$ | 12 | 4 | 6 | 11 | 8 | 12 | 18 | 8 | 21 | |

For example, for $n = 3$ the region corresponding to the channels $\text{BAC}^3(p, q)$ with $\frac{1}{2} < S(p, q) < 1$ (the 3-fold quasi-symmetric channels) represents the 47% of the total area. The last percentage in each row of this table corresponds to the criterion \mathcal{A}_{QS}^n , associated with the n -fold quasi-symmetric channels. By (4), the number of stable regions for $n = 8$ and $n = 9$ is $t_8 = 11$ and $t_9 = 14$ respectively. The percentages represented by these regions (from left to right) are (9.09, 2.58, 3.85, 6.13, 10.54, 8.41, 12.12, 11.28, 7.00, 8.16, 20.84) for $n = 8$ and (7.28, 1.81, 2.58, 3.85, 6.13, 4.49, 6.06, 8.41, 12.12, 11.28, 7.00, 8.16, 4.59, 16.24), for $n = 9$. As we can see, the criterion \mathcal{A}_{QS}^n maximizes the probability among all the criteria for the n -fold channels for $4 \leq n \leq 9$. We have checked also for $n \leq 40$ (see Figure 5 to see graphically the percentages represented by all regions for $n = 40$) and conjecture that this is true for any $n \geq 4$. Let \mathcal{A}_Z^n be the stable criterion for the n -fold BAC nearest to the n -fold Z -channel. We also point out an interesting comparison regarding the sizes of the areas corresponding to the criteria \mathcal{A}_{QS}^n and \mathcal{A}_Z^n . By considering for each n the ratios $R(n)$ and $r(n)$ between the areas corresponding to \mathcal{A}_{QS}^n and \mathcal{A}_Z^n and the average area for this n , it can be observed from our data that $R(n)$ grows (with some very small oscillation) with n linearly (i.e. $R(kn)/R(n)$ approaches k) whereas $r(n)$ gets near to one. As a sample, for $n = 4, 8, 16, 18, 25, 36, 49, 40, 50, 100$ and 200 , the obtained sequences $(n, R(n), r(n))$ are (4, 1.418, 0.966), (8, 2.292, 0.100), (16, 3.908, 0.966), (18, 4.398, 0.980), (25, 5.867, 1.012), (36, 8.299, 0.978), (40, 9.217, 0.983), (50, 11.588, 0.998), (100, 22.559, 0.991) and (200, 45.098, 1.001).

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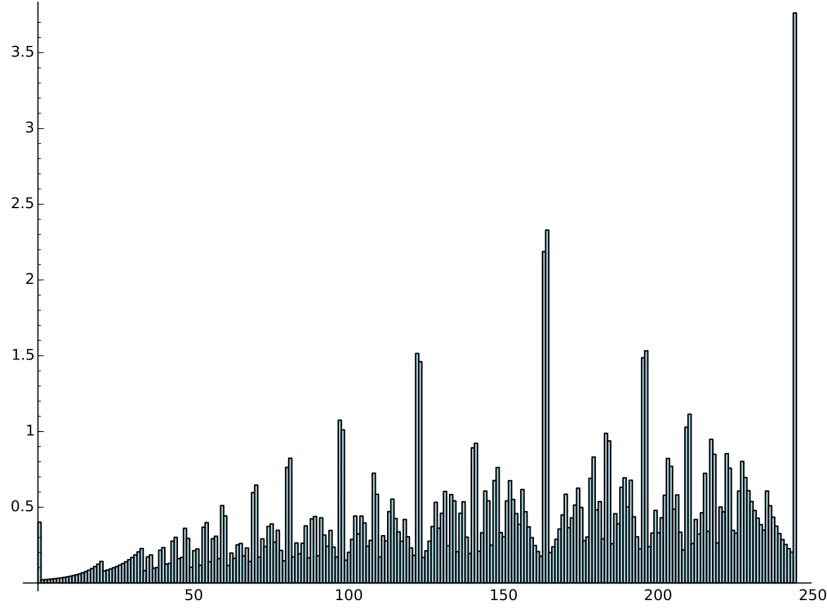


FIGURE 5. The percentages represented by the 245 stable regions for $n = 40$ ordered by its BAC-function values. The rightmost bar corresponds to \mathcal{A}_{QS}^{40} .

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